

# ASYMPTOTIC REPRESENTATION THEOREMS FOR POVERTY INDICES

GANE SAMB LO AND SERIGNE TOUBA SALL

**ABSTRACT.** We set general conditions under which the general poverty index, which summarizes all the available indices, is asymptotically represented with some empirical processes. This representation theorem offers a general key, in most directions, for the asymptotics of the bulk of poverty indices and issues in poverty analysis. Our representation results uniformly hold on a large collection of poverty indices. They enable the continuous measure of poverty with longitudinal data.

In quantitative poverty analysis, poverty indices are the key tools as well as inequality measures. A great number of such indices have been introduced in the literature since the pioneering works of the Nobel Prize winner, Amartya Sen (1976) who first derived poverty measures (see [11]) from an axiomatic point of view. A survey of these indices is to be found in Zheng [13], who also discussed their properties and classified them from an axiomatic point of view.

Statistical asymptotic laws for these indices, particularly asymptotic normality, on which statistical inference on the unknown poverty index may be based, are also of great importance. Recent works which dealt with this, are available in [1], [6], [3] and [2] for instance. These results reveal themselves very powerful and showed real interest in applications. Nevertheless, the indices are studied mostly one by one. In [7], a unified approach is proposed with a general form of the poverty indices, named the General Poverty Index (GPI), including almost all the proposed indices. In [7] and [10], a general asymptotic theory of the GPI is given, based on the so-called Hungarian approximations (see [4] and [5]). Now there is much to do when we deal with longitudinal data. In this case, one has to move from a static approach to a time-dependent one. Moreover, the GPI and a large class of inequality measures, form

---

*Key words and phrases.* Welfare indices, time-dependent L-statistics, functional approximation, empirical process.

classes of L-Statistics indexed by functions.

We aim at giving general tools from which, the functional and time-dependent asymptotic laws of the GPI will be derived. Precisely, we give here functional and time-dependent representation theorems of the class of poverty indices into a functional empirical process, largely described in [12], and a new functional process to be especially handled in [8].

Now, let us make some notation that will permit to clarify the ideas and formulate the problems. We consider a population of individuals, each of which having a random income or expenditure  $Y$ , with distribution function  $G$ . An individual or a household is considered as poor whenever  $Y$  fulfills  $Y \leq Z$ , where  $Z > 0$  is a specified threshold level named the poverty line.

Consider now a random sample  $Y_1, Y_2, \dots, Y_n$  of size  $n$  of incomes, with empirical distribution function  $G_n(y) = n^{-1} \# \{Y_i \leq y : 1 \leq i \leq n\}$ . The number of poor individuals within the sample is then equal to  $Q_n = nG_n(Z)$ . Let also be measurable functions  $A(p, q, z)$ ,  $w(t)$ , and  $d(t)$  of  $p, q \in \mathbb{N}$ , and  $z, t \in \mathbb{R}$  and  $B(q) = \sum_{i=1}^q w(i)$ .

Let finally  $Y_{1,n} \leq Y_{2,n} \leq \dots \leq Y_{n,n}$  be the order statistics of the sample  $Y_1, Y_2, \dots, Y_n$  of  $Y$ . We introduce the following

$$(0.1) \quad J_n = \frac{A(Q_n, n, Z)}{nB(Q_n)} \sum_{j=1}^{Q_n} w(\mu_1 n + \mu_2 Q_n - \mu_3 j + \mu_4) d\left(\frac{Z - Y_{j,n}}{Z}\right),$$

where  $\mu_1, \mu_2, \mu_3, \mu_4$  are constants, as the General Poverty Index (GPI) like in [7] and [10].

We already showed in [10] how to derive from (0.1) the individual poverty measures like the Sen, Kakwani, Foster-Greer-Thorbecke, Thon, Chakravarty ones and some other ones. (See [13], for a definition of such measures). We do not need to return back to this in this paper.

As said previously, our aim is to obtain functional asymptotic laws for the time-dependent GPI. We then need to define our index set in (0.1). Suppose that the functions  $A$ ,  $w$  and  $d$  are in some classes of positive and measurable functions  $\mathcal{C}_i, i = 1, 2, 3$  with these further specifications :  $\mathcal{C}_1$  is a class of functions  $A(p, q, z)$  with  $(p, q, z) \in \mathbb{N}^3$ ,  $\mathcal{C}_2$  of functions  $w(t)$  with  $t \in \mathbb{R}$  and  $\mathcal{C}_3$  of functions  $d(\cdot)$  continuous

and defined on  $[0, 1]$  onto  $[0, 1]$  and bounded by one. The constants vector  $\mu = (\mu_1, \dots, \mu_4)^t$  lies also in some subset  $\mathcal{C}_4$  of  $\mathbb{N}^4$ . We put  $\lambda_0 = (A, w, u) \in \Gamma_0 = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_4$ . In the longitudinal data case, we observe the same households over the time. This leads to the longitudinal observations of  $Y \in C([0, T])$ ,

$$\{Y_1(t), \dots, Y_t(t), 0 \leq t \leq T\},$$

where for each  $t \in [0, T]$ ,  $G_t(\cdot)$  stands for the distribution function of  $Y(t)$ . We consequently use the index

$$\lambda = (A, w, u, t) \in \Gamma = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_4 \times [0, T]$$

and

$$\phi = (A, w, d, u, t) \in \Phi = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \times [0, T].$$

In the time-dependent case, the poverty line  $Z(t)$  may depend on the time and so does the poor headcount denoted as  $Q_n(t)$ . With these notations, our object study becomes

$$J_n(\phi) = \frac{A(Q_n(t), n, Z(t))}{nB(Q_n(t))} \sum_{j=1}^{Q_n(t)} w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) d\left(\frac{Z(t) - Y_{j,n}(t)}{Z(t)}\right).$$

We are now giving our fundamental result, that is the uniform representation of  $\{\sqrt{n}(J_n(\phi) - J(\phi)), \phi \in \Phi\}$  in terms of functional empirical processes, where  $J(\phi)$  is the exact GPI.

In the sequel, we use almost sure limits in outer probability (*a.s.o.p.*), limits to zero in outer probability denoted  $o_P^*(1)$ , following the new theory in [12], when dealing with non-measurable processes. We point out for once that all the limits in this paper are meant as  $n \rightarrow +\infty$ , unless the contrary is specified.

## 1. FUNDAMENTAL THEOREM

Define  $\mathcal{F}_0 = \{f_t : x \rightsquigarrow 1_{(x(t) \leq Z(t))}, t \in [0, T]\}$  as a subset of  $\ell^\infty(C([0, T]))$ , the set of real bounded and continuous functions defined on  $C([0, T])$ . We will consider these general assumptions.

(HP0) There exist  $\beta > 0$  and  $0 < \xi < 1$  such that

$$0 < \beta < \inf_{0 \leq t \leq T} G_t(Z) < \sup_{0 \leq t \leq T} G_t(Z) < \xi < 1.$$

(HP1) The family  $\mathcal{F}_0$  is a  $\mathbb{P}_Y$ -Glivenco-Cantelli class, that is, as  $n \rightarrow \infty$ ,

$$\sup_{t \in [0, T]} |G_{t,n}(Z(t)) - G_t(Z(t))| \rightarrow 0, \text{ a.s.o.p.}$$

where, for any  $t \in [0, T]$  and  $y \in \mathbb{R}$ ,  $G_{t,n}(y) = n^{-1} \sum_{i=1}^n 1_{(Y(t) \leq y)}$ .

(HP2a) There exist a class  $\mathcal{C}$  of functions  $c(s, t)$  of  $(s, t) \in \mathbb{R}^2$  and a class  $\Pi$  of functions  $\pi(s, t)$  of  $(s, t) \in \mathbb{R}^2$ , such that for any  $\lambda = (A, w, \mu, t) \in \Gamma$ , one can find a single function  $h(p, q)$  of  $(p, q) \in \mathbb{N}^2$  such that there exist a function  $c \in \mathcal{C}$  and a function  $\pi \in \Pi$ , all of them independant of  $t \in [0, T]$ , under which we have, as  $n \rightarrow +\infty$ ,

$$\sup_{\lambda \in \Gamma} \max_{1 \leq j \leq Q_n(t)} \left| A(n, Q_n(t)) h^{-1}(n, Q_n(t)) w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) \right. \\ \left. - c(Q_n(t)/n, j/n) \right| = o_P^*(n^{-1/2}).$$

(HP2b) and

$$\sup_{\lambda \in \Gamma} \max_{1 \leq j \leq Q_n(t)} \left| w(j) h^{-1}(n, Q_n(t)) - \frac{1}{n} \pi(Q_n(t)/n, j/n) \right| = o_P^*(n^{-1/2})$$

(H2Pc) There exists a function  $c(u, v)$  of  $(u, v) \in (0, 1)^2$  independent of  $t \in [0, T]$ , such that, as  $n \rightarrow +\infty$ ,

$$\sup_{t \in [0, T]} \max_{1 \leq j \leq Q_n(t)} \left| A(n, Q_n(t)) h^{-1}(n, Q_n(t)) w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) \right. \\ \left. - c(Q_n(t)/n, j/n) \right| = o_P^*(n^{-1/2}).$$

(HP3) The elements of the classes  $\mathcal{C}$  and  $\Pi$  have equi-continuous partial differentials in the sense that (for example for  $c \in \mathcal{C}$ ) :

$$\lim_{(k, l) \rightarrow (0, 0)} \sup_{(x, y) \in (0, 1)^2} \sup_{c \in \mathcal{C}} \left| \frac{\partial c}{\partial y}(x + l, y + k) - \frac{\partial c}{\partial y}(x, y) \right| = 0,$$

and

$$\lim_{(k, l) \rightarrow (0, 0)} \sup_{\beta \leq x \leq \xi, y \in (0, 1)} \sup_{c \in \mathcal{C}} \left| \frac{\partial c}{\partial x}(x + l, y + k) - \frac{\partial c}{\partial x}(x, y) \right| = 0.$$

(HP4) For any  $(c, \pi) \in \mathcal{C} \times \Pi$ , for fixed  $x$ , the functions  $y \rightarrow \frac{\partial c}{\partial y}(x, y)$  and  $y \rightarrow \frac{\partial \pi}{\partial y}(x, y)$  are monotone.

(HP5) For any  $t \in [0, T]$ ,  $G_t$  is strictly increasing, and the functions  $G_t$  are equi-continuous in  $t \in [0, 1]$ .

(HP6) There exist  $H_0 > 0$  and  $H_\infty < +\infty$  such that, for any  $(t, c, \pi, d) \in [0, T] \times \mathcal{C} \times \Pi \times \mathcal{C}_3$ ,

$$H_0 < H_c(\phi) = \int_0^{+\infty} c(G_t(Z), G_t(y)) \gamma(y) dG_t(y) < H_\infty,$$

and

$$H_0 < H_\pi(\phi) = \int_0^{+\infty} \pi(G_t(Z), G_t(y)) e(y) dG_t(y) < H_\infty;$$

where  $\gamma(x) = d(\frac{Z-x}{Z}) 1_{(x \leq Z)}$  and  $e(x) = 1_{(x \leq Z)}$  for  $x \in \mathbb{R}$ . Here and in the sequel, the functions depend, in some way, on  $\phi$  even when we do not specify it or even when we only partially do it.

(HP7) There is a universal constant  $K_0$ , such that there exists  $\delta > 0$ , there exists  $r > 0$  such that

$$(1.1) \quad |s - t| \leq \delta \implies \left| \frac{1}{3} - \mathbb{E}(G_t(Y(t))G_s(Y(s))) \right| \leq K_0 |s - t|^{1+r},$$

Here is our fundamental tool Theorem.

**Theorem 1.** *Suppose that the hypotheses (HP1)-(HP7) hold. Put*

$$J(\phi) = H_c(\phi)/H_\pi(\phi),$$

$$g_t = H_2^{-1}g_c - H_c H_\pi^{-2}g_\pi + K e(f_t(\cdot))$$

with

$$(1.2) \quad g_c(\cdot) = c(G_t(Z), G_t(f_t(\cdot))) \gamma(f_t(\cdot)), \quad g_\pi = \pi(G_t(Z), G_t(f_t(\cdot))) e(f_t(\cdot)),$$

$$(1.3) \quad K(\phi) = H_\pi^{-1}K_c - H_c H_\pi^{-2}K_\pi$$

$$(1.4)$$

$$K_c(\phi) = \int_0^1 \frac{\partial c}{\partial x}(G_t(Z), s) \gamma(G_t^{-1}(s)) ds, \quad K_\pi(\phi) = \int_0^1 \frac{\partial \pi}{\partial x}(G_t(Z), s) e(G_t^{-1}(s)) ds,$$

$$\nu = H_\pi^{-1}\nu_c - H_c H_\pi^{-2}\nu_\pi,$$

where

$$\nu_{c,t}(y) = \frac{\partial c}{\partial x}(G_t(Z), G_t(f_t(y))) \gamma(f_t(y)), \quad \nu_{\pi,t}(y) = \frac{\partial \pi}{\partial x}(G_t(Z), G_t(f_t(y))) e(f_t(y)).$$

Then we have, uniformly in

$$\phi = (A, w, d, \mu, t) \in \Phi = \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 \times [0, T],$$

the following representation :

$$(R) \quad \sqrt{n}(J_n(\phi) - J(\phi)) = \alpha_{t,n}(g_t) + \beta_n(t, \nu_t) + o_P^*(1),$$

with

$$\alpha_{t,n}(g_t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n g_t(Y_j) - \mathbb{E}g(Y_j)$$

and

$$\beta_n(t, \nu_t) = \frac{1}{\sqrt{n}} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\} \nu_t(Y_j).$$

Suppose that (H2Pc) holds in place of (H2Pa) and (HP2b) and the other assumptions are true. Then the representation (R) holds with

$$(RD) \quad K(t) = K_c(t), g_t = g_{c,t} \text{ and } \nu_t = \nu_{c,t}$$

**Remark 1.** The conditions may seem hard to hold. But, for the classical poverty measures, (HP2a), (HP2b) and (HP4) hold. (HP4) also holds for the Kakwani class. The other hypotheses depend on the distribution of  $\{Y(t), t \in [0, T]\}$  and the properties of the poverty lines  $\{Z(t), t \in [0, T]\}$ .

**Remark 2.** With the representation (R),  $\sqrt{n}(J_n(\phi) - J(\phi))$  may be studied in many ways whenever the properties of the processes  $\alpha_{t,n}(g_t)$  and  $\beta_n(t, \nu_t)$  are known as well as their covariance structure. The first is nothing else than the functional process. It is largely studied in [12]. The second  $\beta_n(t, \nu_t)$ , apparently new, is entirely described in [8] as well as its correlation with the functional empirical process. Thus, Formula (R) opens various poverty study fields : using poverty indices with longitudinal data, simultaneous comparison poverty situations with several indices, statistical estimation of lack of decomposability, uniform bootstrapping of poverty indices, etc. Such studies using (R) are underway. As an example, it is used in [9] to get confidence intervals of the relative change in poverty. This method enables to check whether the Millenium Development Goal (MDG) of halving poverty in some time interval is achieved with a probability confidence, say 95%. In the same time, it is showed in [9] how to combine (R) with the results of  $\beta_n(t, \nu_t)$  in [8] to get concluding applications in specific problems.

## 2. PROOFS

Let us begin by giving general considerations. We introduce these notations to be used later in the proofs. First, based on the hypothesis (H5), we may use the following representation. First, define the rank statistic  $R_n(t) = (R_{1,n}(t), \dots, R_{n,n}(t))$  based on  $Y_1(t), \dots, Y_n(t)$  defined by

$$\forall (i, j) \in \{1, \dots, n\}^2, R_{j,n}(t) = i \Leftrightarrow Y_j(t) = Y_{i,n}(t).$$

Since each  $G_t$  is a one-to-one mapping function, we have

$$G_{t,n}(Y_j) = R_{j,n}(t)/n, \text{ a.s.}$$

Secondly, we remark that  $U_1(t) = G_t(Y_1), U_2(t) = G_t(Y_2(t)), \dots$  are independent uniform random variables whenever  $G_t$  is increasing. We will have to consider  $U_{t,n}(\cdot)$  and  $V_{t,n}(\cdot)$ , respectively the empirical distribution and quantile functions based on  $U_1(t), \dots, U_n(t)$ . Denote also  $\alpha_{t,n}(s) = \{\sqrt{n}(U_{t,n}(s) - s), s \in (0, 1)\}$ , the empirical process based on  $Y_1(t), Y_2(t), \dots, Y_n(t)$ ; for  $n \geq 1$ .

Now, by (HP1),

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \sup_{y \in \mathbb{R}} |G_{t,n}(y) - G_t(y)| = 0, \text{ a.s.o.p}$$

By (HP2), uniformly in  $\phi \equiv (A, w, \mu, d, t) \in \Phi = C_1 \times C_2 \times C_3 \times C_4 \times [0, T]$ ,

$$J_n = \frac{A(n, Q_n(t))/h(n, Q_n(t), j)}{n \sum_{j=1}^{Q_n(t)} w(j)/h(n, Q_n(t), j)} \sum_{j=1}^{Q_n(t)} w(\mu_1 n + \mu_2 Q_n(t) - \mu_3 j + \mu_4) d\left(\frac{Z - Y_{j,n}(t)}{Z}\right).$$

$$(2.1) \quad = (J_n(c) + o_P^*(n^{-1/2})) / (J_n(\pi) + o_P^*(n^{-1/2})),$$

where

$$(2.2) \quad J_n(c) = \frac{1}{n} \sum_{j=1}^n c(G_{t,n}(Z), G_{t,n}(Y_j)) \gamma(Y_j(t)),$$

and

$$(2.3) \quad J_n(\pi) = \frac{1}{n} \sum_{j=1}^n \pi(G_{t,n}(Z), G_{t,n}(Y_j)) e(Y_j).$$

Now we have

$$J_n(c) = \int_0^1 c(G_{t,n}(Z), U_{t,n}(V_{t,n}(s))) \gamma(G^{-1}(V_{t,n}(s))) ds,$$

and

$$H_c(\phi) = \int_0^1 c(G_t(Z), s) \gamma(G^{-1}(s)) ds.$$

Then

$$\begin{aligned} J_n(1) &= \frac{1}{n} \sum_{j=1}^n c(G_t(Z), G_t(Y_j)) \gamma(Y_j(t)) \\ &+ \frac{1}{n} \sum_{j=1}^n \{c(G_{t,n}(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_{t,n}(Y_j))\} \gamma(Y_j(t)) \end{aligned}$$

$$+\frac{1}{n} \sum_{j=1}^n \{c(G_t(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_t(Y_j))\} \gamma(Y_j(t)).$$

Set,

$$D_1 = \frac{1}{n} \sum_{j=1}^n \{c(G_{t,n}(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_{t,n}(Y_j))\} \gamma(Y_j(t)),$$

and

$$D_2 = \frac{1}{n} \sum_{j=1}^n \{c(G_t(Z), G_{t,n}(Y_j)) - c(G_t(Z), G_t(Y_j))\} \gamma(Y_j(t)) \equiv D_1 + D_2.$$

First, we have by using the classical representations,

$$D_1 = \int_0^1 \{c(G_{t,n}(Z), U_{t,n}(V_{t,n}(s))) - c(G_t(Z), U_{t,n}(V_{t,n}(s)))\} \gamma(G^{-1}(V_{t,n}(s))) ds$$

(2.4)

$$= (G_{t,n}(Z) - G_t(Z)) \int_0^1 \left\{ \frac{\partial c}{\partial x}(\zeta_n(t, Z), U_{t,n}(V_{t,n}(s))) \gamma(G^{-1}(V_{t,n}(s))) \right\} ds$$

where  $\zeta_n(t, Z)$  lies between  $G_t(Z)$  and  $G_{t,n}(Z)$ . We want now to prove that integral factor in (2.4) tends to  $K_c(\phi)$  uniformly in  $\phi$ . We shall use a method that will be repeated throughout the paper. First, we remark that this is performed in  $s \in (V_{t,n}(s) \leq G_t(Z))$ . Since  $V_{t,n}(s) \rightarrow s$  uniformly in  $(t, s) \in (0, T) \times (0, 1)$ , in outer probability, we have for any  $\epsilon > 0$ ,  $0 < \xi + \epsilon < 1$ ,  $(V_{t,n}(s) \leq G_t(Z)) \subseteq (s \leq G_t(Z) + \epsilon)$ , uniformly in  $(t, s) \in (0, T) \times (0, 1)$ , with outer probability greater or equal to  $1 - \epsilon$  (denoted  $w.p.1 - \epsilon$ ), for large values of  $n$ . Thus for such  $n$ 's, we have, uniformly in  $(t, s) \in (0, T) \times (0, 1)$ ,  $w.p.1 - \epsilon$ ,

$$\begin{aligned} & \left| \int_0^1 \frac{\partial c}{\partial x}(\zeta_n(t, Z), U_{t,n}(V_{t,n}(s))) \gamma(G_t^{-1}(V_{t,n}(s))) ds \right. \\ & \quad \left. - \int_0^1 \frac{\partial c}{\partial x}(G_t(Z), s) d\left(\frac{Z - G_t^{-1}(s)}{Z}\right) e(G_t^{-1}(V_{t,n}(s))) ds \right| \\ (2.5) \quad & \leq \int_0^{\xi + \epsilon} \left| \frac{\partial c}{\partial x}(\zeta_n(t, Z), U_{t,n}(V_{t,n}(s))) d\left(\frac{Z - G_t^{-1}(V_{t,n}(s))}{Z}\right) \right. \\ & \quad \left. - \frac{\partial c}{\partial x}(G_t(Z), s) d\left(\frac{Z - G_t^{-1}(s)}{Z}\right) \right| e(G_t^{-1}(V_{t,n}(s))) ds \end{aligned}$$

This latter tends uniformly in  $\phi$  to zero since  $\frac{\partial c}{\partial x}(\cdot, \cdot)$  is uniformly continuous on  $[0, \xi + \epsilon]$  in the sense described in (HP3). By letting  $\epsilon \rightarrow 0$ ,



we get the convergence of (2.5) to zero, uniformly in  $\phi$ , in outer probability. Remark further that

$$(2.6) \quad \left| \int_0^1 \frac{\partial c}{\partial x}(G_t(Z), s) d\left(\frac{Z - G_t^{-1}(s)}{Z}\right) e(G_t^{-1}(V_{t,n}(s))) ds - K_c \right| \\ \leq \int_0^1 \left| \frac{\partial c}{\partial x}(G_t(Z), s) d\left(\frac{Z - G_t^{-1}(s)}{Z}\right) \right| |e(G_t^{-1}(V_{t,n}(s))) - e(G_t^{-1}(s))| ds.$$

Now  $|e(G_t^{-1}(V_{t,n}(s))) - e(G_t^{-1}(s))|$  is the indicator function of the symmetrical difference of the sets  $(V_{t,n}(s) \leq G(Z))$  and  $(s \leq G_t(Z))$ . And we have, for large values of  $n$ ,  $w.p.1 - \epsilon$

$$(2.7) \quad (V_{t,n}(s) \leq G_t(Z)) \Delta (s \leq G_t(Z)) \subseteq (G_t(Z) - \epsilon \leq s < G_t(Z)) + (G_t(Z) \leq s \leq G_t(Z) + \epsilon).$$

This and the uniform boundedness in  $\phi$ , (say by  $M$ ), of the functions  $\frac{\partial c}{\partial x}(G_t(Z), s) d\left(\frac{Z - G_t^{-1}(s)}{Z}\right)$ , due its uniform continuity on  $[0, G(Z) + \epsilon_0]$  in the sense of (HP3), imply that the second member in (2.6) is, uniformly in  $\phi$ , less than  $2M\epsilon$ ,  $w.p.1 - \epsilon$ . By letting  $\epsilon \rightarrow 0$ , we finally get the convergence of the integral in (2.6), uniformly in  $\phi$ , to  $K_c$ . This kind of arguments will be used in the sequel whithout further details. It follows that

$$(2.8) \quad D_1 = K_c(G_n(Z) - G(Z))/\sqrt{n} + o_p^*(n^{-1/2}),$$

uniformly in  $\phi$ . Next

$$(2.9) \quad D_2 = \frac{1}{n} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\} \frac{\partial c}{\partial y}(G_t(Z), \zeta_{t,n}(j)) \gamma(Y_j(t));$$

where  $\zeta_{t,n}(j)$  lies between  $G_{t,n}(Y_j(t))$  and  $G_t(Y_j(t))$ . Now, denoting  $I_n = [G_{t,n}(Y_j(t)) \wedge G_t(Y_j(t)), G_{t,n}(Y_j(t)) \vee G_t(Y_j(t))]$ , we have to show that

$$(2.10) \quad \max_{1 \leq j \leq n} \sup_{t \in [0, T]} \sup_{(x, y) \in I_n^2} \left| \frac{\partial c}{\partial y}(G_t(Z), x) - \frac{\partial c}{\partial y}(G_t(Z), y) \right| \gamma(Y_j(t)) \rightarrow_{P^*} 0,$$

as  $n \rightarrow +\infty$ . The boundedness of  $\gamma(\cdot)$ , the equi-continuity of  $\frac{\partial c}{\partial y}$  and (HP1) establish (2.10). By (HP7) and Theorem 2 in [8], the process  $B_n^*(\cdot)$  defined by:

$$\{B_n^*(t), 0 \leq t \leq T\} = \left\{ \frac{1}{\sqrt{n}} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\}, 0 \leq t \leq T \right\}$$

converges to a Gaussian process  $\{G_0(t), 0 \leq t \leq T\}$  in  $\ell^\infty([0, T])$  under the hypotheses. Thus,

$$\sqrt{n}D_2 = \frac{1}{\sqrt{n}} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\} \frac{\partial c}{\partial y}(G_t(Z), G_t(Y_j(t))) \gamma(Y_j(t)) + o_P^*(1),$$

uniformly in  $\phi \in \Phi$ . We conclude that

$$\begin{aligned} \sqrt{n}(J_n(c) - H_c) &= \frac{1}{\sqrt{n}} \sum g(Y_j) - Eg(Y_j) + K_1 \sqrt{n}(G_{t,n}(Z) - G(Z)) \\ &\quad + \frac{1}{\sqrt{n}} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\} \nu_{c,t}(y) + o_P^*(1) \end{aligned}$$

Now, we have to handle in the same lines to get for  $J_n(\pi)$

$$\begin{aligned} \sqrt{n}(J_n(\pi) - H_\pi) &= \frac{1}{\sqrt{n}} \sum g_\pi(Y_j) - Eg_\pi(Y_j) + K_\pi \sqrt{n}(G_{t,n}(Z) - G(Z)) \\ &\quad + \frac{1}{\sqrt{n}} \sum \{G_{t,n}(Y_j(t)) - G_t(Y_j(t))\} \nu_{\pi,t}(Y_j) + o_P^*(1). \end{aligned}$$

Remark that  $H_c(\phi)$  and  $H_\pi(\phi)$  are uniformly bounded. Then we arrive at the representation (R).

Now, when (HP2c) holds for  $h(n, Q) = B(n, Q)$ , the quotient of (2.1) is one. This leads to the representation (RD) only based on that of  $J_n(c)$ .

## REFERENCES

- [1] Barrett G. and Donald, S.(2000). Statistical Inference with Generalized Gini Indices of Inequality and Poverty. Available at : (<http://www.eco.utexas.edu/~donald/research/genginir.pdf>)
- [2] Bishop J.A., Chow K.V, and Zheng Z.(1995). Statistical Inference and Decomposable Poverty Measures. *Bulletin of Economic Research*, **47**, pp.329-340.
- [3] Bishop J.A., Formby J.P., Zheng Z.(1997). Statistical Inference and the Sen Index of Poverty. *International Economic Review*, 38, (2), pp. 381-387.
- [4] Csörgö, M., Csörgö, S., Horvath, L. and Mason, M. (1986). Weighted empirical and quantile processes. *Ann. Probab.*, 14, 31-85.
- [5] Csörgö M. and Horvath, L.(1986). Approximations of weighted empirical and quantile processes. *Statistics and Probability letters*, 4, 275-280.

- [6] Davidson R. and Duclos Y.(2000). Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality. *Econometrica*, 68 (6), pp.1435-1464.
- [7] Lo, G.S., Sall. S. and Seck, C.T.(2009). Une théorie asymptotique des indicateurs de pauvreté. *C.R. Math. Rep. Acad. Sci. Canada*, 45-52, 31 (2)
- [8] Lo, G.S.(2009). On some residual stochastic process. <http://www.ufrsat.org/perso/gslo/empstopro.pdf>
- [9] Lo, G.S, and Sall S.T.(2009). The Asymptotic Laws of the Time-dependent General Poverty Index and Applications.  
Available at : <http://www.ufrsat.org/perso/gslo/sall-lo00210.pdf>
- [10] Sall S. T. and Lo G. S.(2009). The asymptotic theory of the Kakwani class of poverty measures. *African Diaspora Jour. Math.*, (1), 54-67.
- [11] Sen A.K.(1976). Poverty: An Ordinal Approach to Measurement. *Econometrica*, 44, 219-231.
- [12] van der Vaart A.W and Wellner J.A.(1996). *Weak Convergence and Empirical Processes With Applications to Statistics*. Springer, New-York.
- [13] Zheng, B.(1997). Aggregate Poverty Measures. *Journal of Economic Surveys*, 11 (2), 123-162.

LSTA, UNIVERSITÉ PARIS VI. FRANCE AND LERSTAD, UNIVERSITÉ GASTON BERGER, SENE-  
GAL.

*E-mail address:* `ganesamblo@ufrsat.org`

FASTEF, UNIVERSITÉ CHEIKH ANTA DIOP AND LERSTAD, UNIVERSITÉ GASTON BERGER DE  
SAINT-LOUIS.

*E-mail address:* `stsall@ufrsat.org`